MAJORANA SPINORS

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Contents

These notes arose as an attempt to conceptualise the 'symplectic Majorana–Weyl condition' in 5+1 dimensions; but have turned into a general discussion of spinors. Spinors play a crucial role in supersymmetry. Part of their versatility is that they come in many guises: 'Dirac', 'Majorana', 'Weyl', 'Majorana–Weyl', 'symplectic Majorana', 'symplectic Majorana–Weyl', and their 'pseudo' counterparts. The traditional physics approach to this topic is a mixed bag of tricks using disparate aspects of representation theory of finite groups. In these notes we will attempt to provide a uniform treatment based on the classification of Clifford algebras, a work dating back to the early 60s and all but ignored by the theoretical physics community. Recent developments in superstring theory have made us re-examine the conditions for the existence of different kinds of spinors in spacetimes of arbitrary signature, and we believe that a discussion of this more uniform approach is timely and could be useful to the student meeting this topic for the first time or to the practitioner who has difficulty remembering the answer to questions like "*when do symplectic Majorana–Weyl spinors exist?*"

The notes are organised as follows. The first section discusses real and quaternionic representations (of a group, say) in terms of complex representations with extra structure and in particular makes the connection to the to the existence of complex bilinear forms. Section 2 recapitulates some of the discussion in Section 1 in terms of matrices. This is useful for explicit calculations as well as to bring home some of the abstract discussion of Section 1. Section 3 discusses Clifford algebras and their representations. It includes the classification of real Clifford algebras as well as a method to build explicit realisations of Clifford algebras in higher dimensions in terms of Pauli matrices; although these realisations are not always the most useful for the problem at hand. Section 4 makes contact with the physics treatment of Clifford algebras; in particular it contains some brief discussion of complex Clifford algebras and serves to contrast the traditional approach to the Majorana condition with the approach advocated here. Section 5 contains lots of examples. Two more sections are planned: section 6 will discuss the theory behind the possible inner products for spinors, whereas section 7 will contain many examples and some applications to physics.

Two remarks before we start. Firstly, these notes are very preliminary. If this were software it would be a pre-release alpha version. In particular, they are not yet meant for widespread circulation. And lastly, a remark on notation. True to tradition, physicists and mathematicians do not quite agree on what to call representations of Clifford algebras and their associated Spin groups: whereas physicists found it useful to confuse the two and use the word 'spinor' interchangeably for both, mathematicians, in their infinite wisdom, introduced the concept of 'pinor' to denote irreducible representations of the Clifford algebra (in fact, of the Pin group) leaving 'spinor' to denote the irreducible representations of the Spin group. I cannot bring myself to choose one nomenclature over the other: while I find it confusing to use spinor for both, I cannot divorce myself of the physics notation which these notes hope to reconcile. I have therefore decided, when faced with a conflict, to use sans-serif type to distinguish the physics usage from the more uniform usage I have tried to adhere to. I follow closely the treatment in [LM89] and in particular [Har90]. Other references of interest are [Wan89], [KT83] and [vN83].

1. Complex, real and quaternionic representations

In this section we try to understand real and quaternionic representations as complex representations with extra structure. The extra structure will manifest itself in the existence of nondegenerate invariant complex-bilinear forms. Throughout these notes all vector spaces and representations are finite-dimensional.

Fix once and for all a group G throughout the remainder of this section. We are interested ultimately in real semisimple Lie groups which are not necessarily compact (e.g., $\mathrm{Spin}(5,1)$). All we will assume is that every representation has an invariant non-degenerate sesquilinear form. We will call such forms "hermitian" but no assumption is made on positive-definiteness.

Definition 1. *Let V be a vector space over $\mathbb{k} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*[1] *By a* hermitian form *on V we mean an $\mathbb{R}$-bilinear form $\langle -,- \rangle : V \times V \to \mathbb{k}$ such that for all $v_1, v_2 \in V$ and $\lambda \in \mathbb{k}$:*

- $\langle v_1, v_2\lambda \rangle = \langle v_1, v_2 \rangle \lambda$; *and*
- $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$.

In particular this implies that $\langle v_1\lambda, v_2 \rangle = \overline{\lambda}\langle v_1, v_2 \rangle$.

Let V be a representation of G over $\mathbb{k} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, by which we mean a representation of G in terms of $\mathbb{k}$-linear maps. If G is a finite group, a compact Lie group or more generally a semisimple Lie group, then V has a G-invariant nondegenerate hermitian form in the above sense. For G finite or compact Lie, this can be proven as follows: pick any positive-definite hermitian form and average over the group to obtain a form which is G-invariant. Because we started with a positive-definite form, the averaged hermitian form remains positive-definite as it is a "sum" of positive-definite forms. If G is semisimple but noncompact, the result

[1] By a quaternionic vector space we mean a right $\mathbb{H}$-module; that is, quaternions act on the right. In this way they don't interfere with quaternionic matrices acting on the left.

still holds, but the proof is more involved. The same is true for representations of Clifford algebras, because a Clifford algebra is (almost) the group algebra of a finite group. Since the semisimple noncompact groups we will be dealing with are Spin groups of some Clifford algebra, their representations will admit an invariant hermitian metric.

Definition 2. *Let V be a complex vector space. We say that a linear map $\varphi : V \to V$ is a* real *(resp.* quaternionic*) structure if φ obeys the following two conditions:*

- *φ is conjugate linear: $\varphi(\lambda v) = \overline{\lambda}\varphi(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$; and*
- *$\varphi^2 = \mathbb{1}$ (resp. $\varphi^2 = -\mathbb{1}$).*

Lemma 1. *Let V be a complex vector space and $c : V \to V$ be a real structure. Then $V = V_+ \oplus V_-$ where $V_\pm$ are isomorphic real vector spaces; equivalently $V \cong \mathbb{C} \otimes V_+$.*

Proof. Since c obeys $c^2 = \mathbb{1}$, its has eigenvalues ± 1. Let $V = V_+ \oplus V_-$ denote the decomposition of V into eigenspaces of c. Because c is not complex linear but only conjugate linear, $V_\pm$ are not complex subspaces but only real subspaces. In fact, if $v \in V_+$ so that $c(v) = v$, $iv \in V_-$: $c(iv) = -ic(v) = -iv$. Hence $i : V_+ \to V_-$ is an isomorphism, and $V = V_+ \oplus iV_+ \cong \mathbb{C} \otimes V_+$. □

A real structure in a complex vector space is nothing but a notion of complex conjugation. Every complex vector space V admits many real structures. They are constructed in the following way. Let (v_i) be a complex basis for V. Let V_0 denote the *real* vector space spanned by (v_i). Then define $c : V \to V$ as follows: $c(v_i) = v_i$ and extend to all of V conjugate linearly: $c(\sum_i \lambda_i v_i) = \sum_i \overline{\lambda_i} v_i$.

On the other hand a quaternionic structure in a complex vector space V, as the name suggests, allows us to define a left action of $\mathbb{H}$ on V. Let $J : V \to V$ be a quaternionic structure. Then $J^2 = -\mathbb{1}$ and $Ji = -iJ$. Therefore if $q = a + bj$, $a, b \in \mathbb{C}$, is a quaternion and $v \in V$, we can define $v\, q = av + bJ(v)$. Unlike real structures, not every complex vector space admits a quaternionic structure: its complex dimension must be even. (Compare with the notion of a complex structure in a real vector space.)

Definition 3. *Let V be a complex representation of G. We say that V is of* real *(resp.* quaternionic*) type if V possesses a G-invariant real (resp. quaternionic) structure.*

It turns out that the existence of real or quaternionic structures is intimately related to the existence of nondegenerate complex bilinear forms.

Theorem 1. *A complex representation V of G is of real (resp. quaternionic) type if and only if V admits a nondegenerate symmetric (resp. antisymmetric) G-invariant complex bilinear form $B : V \times V \to \mathbb{C}$.*

Proof. Let $B : V \times V \to \mathbb{C}$ be given satisfying the following conditions:

- B is nondegenerate, complex bilinear and G-invariant; and
- $B(v_1, v_2) = \varepsilon B(v_2, v_1)$ where $\varepsilon = \pm 1$.

Choose a G-invariant hermitian form $\langle -, - \rangle$ on V and define $\varphi : V \to V$ by

$$B(v_1, v_2) = \langle \varphi(v_1), v_2 \rangle \ , \qquad \text{for all } v_1, v_2 \in V.$$

It follows that φ is conjugate linear, G-invariant and an isomorphism. (The proof is left as an exercise for the reader.) We would like to use φ to define the real or quaternionic structure that we are after, but for this φ^2 should be $\varepsilon\mathbb{1}$. It will turn out that this is not true, but will be true after rescaling. Using the symmetry properties of B we see that:

$$\langle \varphi(v_1), v_2 \rangle = B(v_1, v_2) = \varepsilon B(v_2, v_1) = \varepsilon \langle \varphi(v_2), v_1 \rangle = \varepsilon \overline{\langle v_1, \varphi(v_2) \rangle} \ .$$

Applying this twice we find that

$$\langle\varphi^2(v_1), v_2\rangle = \varepsilon\overline{\langle\varphi(v_1), \varphi(v_2)\rangle} = \langle v_1, \varphi^2(v_2)\rangle \ .$$

This implies that the form $\langle\langle -, -\rangle\rangle : V \times V \to \mathbb{C}$ defined by

$$\langle\langle v_1, v_2\rangle\rangle = \varepsilon\langle v_1, \varphi^2(v_2)\rangle$$

is hermitian and positive-definite. That is, $\langle\langle v_1, v_2\rangle\rangle = \overline{\langle\langle v_2, v_1\rangle\rangle}$ and $\langle\langle v, v\rangle\rangle \geq 0$. This implies that the operator $\mu = \varepsilon\varphi^2$ has positive real eigenvalues. We now try to rescale μ in such a way that the resulting operator is $\mathbb{1}$. Because φ is conjugate linear, μ is complex linear, whence V can be split into a direct sum of complex eigenspaces of μ:

$$V = \bigoplus_{\substack{\lambda\in\mathbb{R}\\ \lambda>0}} V_\lambda \ .$$

The action of both G and φ preserve each V_λ. Now define $\nu : V \to V$ by letting $\nu = \frac{1}{\sqrt{\lambda}}\mathbb{1}$ on V_λ. This makes sense since λ is a positive real number. Notice that $\nu\varphi = \varphi\nu$ and that $\mu\nu^2 = \mathbb{1}$. Also ν is G-invariant since it is a scalar on each G-invariant subspace V_λ. Then we simply define $\mathcal{J} = \nu\varphi$. It is G-invariant, conjugate linear since φ is, and it obeys $\mathcal{J}^2 = \nu\varphi\nu\varphi = \nu^2\varphi^2 = \varepsilon\mathbb{1}$. Thus $\mathcal{J}$ is the required structure map.

Conversely, suppose that V has a G-invariant structure map $\mathcal{J} : V \to V$ satisfying $\mathcal{J}^2 = \varepsilon\mathbb{1}$. We want to show that V has the required bilinear form. If $\varepsilon = 1$, then $V \cong \mathbb{C} \otimes V_+$ where V_+ is a real representation of G. By our hypothesis on G, V_+ admits a G-invariant symmetric nondegenerate $\mathbb{R}$-bilinear form. We can extend this by complex linearity to a nondegenerate G-invariant symmetric $\mathbb{C}$-bilinear form on V. This is the required bilinear form. On the other hand, if $\varepsilon = -1$, then V becomes a quaternionic vector space with $j \in \mathbb{H}$ acting via $\mathcal{J}$. By our assumption on G, V carries a G-invariant nondegenerate hermitian form $\langle -, -\rangle$ with values in $\mathbb{H}$. Let us write for all $v_1, v_2 \in V$

$$\langle v_1, v_2\rangle = H(v_1, v_2) + jB(v_1, v_2) \ ,$$

where H and B are complex-valued and G-invariant. We now use the fact that $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$, to show that H is hermitian and B is antisymmetric:

$$\begin{aligned} H(v_1, v_2) + jB(v_1, v_2) &= \langle v_1, v_2\rangle \\ &= \overline{\langle v_2, v_1\rangle} \\ &= \overline{H(v_2, v_1)} + \overline{jB(v_2, v_1)} \\ &= \overline{H(v_2, v_1)} + \overline{B(v_2, v_1)}\bar{j} \\ &= \overline{H(v_2, v_1)} - \overline{B(v_2, v_1)}j \\ &= \overline{H(v_2, v_1)} - jB(v_2, v_1) \ . \end{aligned}$$

Similarly, using that $\langle\lambda v_1, v_2\rangle = \overline{\lambda}\langle v_1, v_2\rangle$ for $\lambda \in \mathbb{H}$, we find that B is $\mathbb{C}$-bilinear. Indeed, if $\lambda \in \mathbb{C}$:

$$\begin{aligned} H(\lambda v_1, v_2) + jB(\lambda v_1, v_2) &= \langle\lambda v_1, v_2\rangle \\ &= \overline{\lambda}\langle v_1, v_2\rangle \\ &= \overline{\lambda}H(v_1, v_2) + \overline{\lambda}jB(v_1, v_2) \\ &= \overline{\lambda}H(v_1, v_2) + j\lambda B(v_1, v_2) \ . \end{aligned}$$

Finally we must show that B is nondegenerate. Assume that $v_0 \in V$ is such that for all $v \in V$, $B(v_0, v) = 0$. Then for all v, $\langle v_0, v\rangle = H(v_0, v) \in \mathbb{C}$. But

$\langle v_0, jv\rangle = j\langle v_0, v\rangle$, whence $\langle v_0, v\rangle = 0$. Since $\langle -,-\rangle$ is nondegenerate, $v_0 = 0$. Hence B is nondegenerate. ☐

Corollary 1. *Let G_1 and G_2 be two groups of the kind discussed in these notes. For $i = 1, 2$, let V_i be a complex representation of G_i of quaternionic type. Then the tensor product $V = V_1 \otimes V_2$ is a complex representation of $G = G_1 \times G_2$ of real type.*

Proof. Let B_i denote the nondegenerate antisymmetric bilinear form on V_i whose existence is guaranteed by the theorem. On $V = V_1 \otimes V_2$ define B by $B(v_1 \otimes v_2, w_1 \otimes w_2) = B_1(v_1, w_1)B_2(v_2, w_2)$. Then B is G-invariant, nondegenerate, $\mathbb{C}$-bilinear and symmetric. By the theorem V is of real type. ☐

2. Some basis-dependent formulae

Since the previous section may have been a little too abstract, let us exhibit some of the relevant formulae after having chosen a basis. If V is a vector space over $\mathbb{k}$, a choice of basis is equivalent to an isomorphism $V \cong \mathbb{k}^n$ for some $n = \dim_{\mathbb{k}} V$. Under such an isomorphism, a vector $v \in V$ is represented by a n-tuple $\boldsymbol{v}$ of elements of $\mathbb{k}$ and $\mathbb{k}$-linear transformations will be represented by $n \times n$ matrices with entries in $\mathbb{k}$. In particular if V is a representation of G, to every element $g \in G$ there corresponds an invertible matrix $\boldsymbol{g}$.

Other objects are also represented by matrices. Let $\langle -,-\rangle$ be a hermitian form. Relative to our chosen basis, it is represented by a hermitian matrix $\boldsymbol{A}$:

$$\langle u, v\rangle = \overline{\boldsymbol{u}}^t \cdot \boldsymbol{A} \cdot \boldsymbol{v} \ ,$$

where $\overline{\boldsymbol{A}}^t = \boldsymbol{A}$. Notice that with these conventions

$$\langle u\lambda, v\rangle = \overline{\lambda}\langle u, v\rangle \ , \quad \langle u, v\lambda\rangle = \langle u, v\rangle\lambda \quad \text{and} \quad \langle u, v\lambda\rangle = \overline{\langle v, u\rangle} \ .$$

If G leaves $\langle -,-\rangle$ invariant, then for all $g \in G$,

$$\overline{\boldsymbol{g}}^t \cdot \boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{A} \ .$$

Similarly, a $\mathbb{k}$-bilinear form B is represented by a matrix $\boldsymbol{B}$:

$$B(u, v) = \boldsymbol{u}^t \cdot \boldsymbol{B} \cdot \boldsymbol{v} \ ,$$

which is symmetric or antisymmetric if B is; that is, if $B(u, v) = \varepsilon B(v, u)$, then $\boldsymbol{B}^t = \varepsilon \boldsymbol{B}$ accordingly. Again if G preserves B, then for all $g \in G$,

$$\boldsymbol{g}^t \cdot \boldsymbol{B} \cdot \boldsymbol{g} = \boldsymbol{B} \ .$$

Notice however that although these inner products are represented by matrices, they do not transform like linear transformations under a change of basis. That means in particular, that even when a linear transformation and a bilinear form, say, may agree as matrices in a given basis, this is *not* an invariant statement. Care should be exercised.

How about the real and quaternionic structures? Let $V \cong \mathbb{C}^n$ be a complex vector space with a chosen hermitian metric $\langle -,-\rangle$ represented by a hermitian matrix $\boldsymbol{A}$. Let B denote the nondegenerate $\mathbb{C}$-bilinear form represented by a matrix $\boldsymbol{B}$ obeying $\boldsymbol{B}^t = \varepsilon \boldsymbol{B}$. Let $\mathfrak{J} : V \to V$ denote the associated structure map. Because $\mathfrak{J}$ is only conjugate linear, it will not be represented by an $n \times n$ complex matrix. Therefore we will have to work with an underlying real basis. Any complex vector space V of complex dimension n can be understood as a real vector space $V_{\mathbb{R}}$ of real dimension $2n$ with a linear map $I : V_{\mathbb{R}} \to V_{\mathbb{R}}$ obeying $I^2 = -\mathbb{1}$. Such a map is called a complex structure. Working in a real basis for V means working with a basis for $V_{\mathbb{R}}$; that is, an isomorphism $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$. Given a $\mathbb{C}$-basis (v_j) for V, a natural $\mathbb{R}$-basis is given by (v_j, iv_j). In this basis the complex structure is represented by the $(2n\times 2n)$ real matrix

$$\boldsymbol{I}_{\mathbb{R}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} ;$$

whereas the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are represented by

$$\boldsymbol{A}_{\mathbb{R}} = \begin{pmatrix} \boldsymbol{A} & -i\boldsymbol{A} \\ i\boldsymbol{A} & \boldsymbol{A} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B}_{\mathbb{R}} = \begin{pmatrix} \boldsymbol{B} & i\boldsymbol{B} \\ i\boldsymbol{B} & -\boldsymbol{B} \end{pmatrix} ,$$

We are finally in a position to describe the matrix representing the real/quaternionic structure. By definition, $B(u,v) = \langle \mathcal{J}(u), v\rangle$. Taking the conjugate of this identity and using the hermiticity of $\langle -,-\rangle$ and the ε-symmetry of B, we see that $\langle v, \mathcal{J}(u)\rangle = \varepsilon\overline{B(v,u)}$. In the chosen basis, this identity becomes

$$\overline{\boldsymbol{v}}^t \cdot \boldsymbol{A} \cdot \boldsymbol{J} \cdot \boldsymbol{u} = \varepsilon\overline{\boldsymbol{v}^t \cdot \boldsymbol{B} \cdot \boldsymbol{u}} .$$

Introducing the real matrices defined above and using the fact that the identity holds for all $\boldsymbol{u}$ and $\boldsymbol{v}$, we find that:

$$\boldsymbol{A}_{\mathbb{R}} \cdot \boldsymbol{J}_{\mathbb{R}} = \varepsilon\overline{\boldsymbol{B}}_{\mathbb{R}} ,$$

where we have used the fact that in a real basis $\overline{\boldsymbol{u}} = \boldsymbol{u}$. This equation determines $\boldsymbol{J}_{\mathbb{R}}$ uniquely in terms of $\boldsymbol{A}_{\mathbb{R}}$ and $\boldsymbol{B}_{\mathbb{R}}$ because $\boldsymbol{A}_{\mathbb{R}}$ and $\boldsymbol{B}_{\mathbb{R}}$ are invertible by hypothesis. Moreover, because $\mathcal{J}$ is a conjugate linear map, the matrix $\boldsymbol{J}_{\mathbb{R}}$ anticommutes with the complex structure $\boldsymbol{J}_{\mathbb{R}}\cdot\boldsymbol{I}_{\mathbb{R}} = -\boldsymbol{I}_{\mathbb{R}}\cdot\boldsymbol{J}_{\mathbb{R}}$. This constraints $\boldsymbol{J}_{\mathbb{R}}$ to take the following form:

$$\boldsymbol{J}_{\mathbb{R}} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \\ \boldsymbol{J}_2 & -\boldsymbol{J}_1 \end{pmatrix} ,$$

where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the matrices representing the real and imaginary parts of the map $\mathcal{J}$, respectively. Using the block forms of $\boldsymbol{A}_{\mathbb{R}}$ and $\boldsymbol{B}_{\mathbb{R}}$ and using the fact that $\boldsymbol{A}$ is hermitian we find the following expression for $\boldsymbol{J} \equiv \boldsymbol{J}_1 + i\boldsymbol{J}_2$:

$$\boxed{\boldsymbol{J} = \left(\boldsymbol{B}\cdot\boldsymbol{A}^{-1}\right)^t} . \tag{1}$$

3. Clifford algebras and their spinors

Our ultimate aim in these notes it to apply the preceding discussion in the context of representations of complex Clifford algebras. It is in this light that the concept of Majorana spinors makes the most mathematical sense (at least to me). But before doing so, we have to collect a few facts about real Clifford algebras and their representations. It is beyond the scope of these notes to give a detailed account of this topic, so we will content ourselves with mentioning some facts.

Given a real vector space E and a quadratic form q defined on it, there is associated a Clifford algebra $\mathrm{C}\ell(E,q)$. When the vector space is d-dimensional and the quadratic form has signature (s,t) with $d = s+t$, the resulting Clifford algebra is known as $\mathrm{C}\ell(s,t)$. A model for this (E,q) is given by $E = \mathbb{R}^d$ with the quadratic form given by:

$$q(x) = x_1^2 + x_2^2 + \cdots + x_s^2 - x_{s+1}^2 - x_{s+2}^2 - \cdots - x_{s+t}^2 ,$$

for $x = (x_1, x_2, \ldots, x_d)$. The Clifford algebra $\mathrm{C}\ell(s,t)$ is generated by elements Γ_a which obey the identities (notice the sign!):

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = -2\eta_{ab}\mathbb{1} , \tag{2}$$

where η_{ab} are the entries of the matrix representing the quadratic form. Clifford algebras are important because we can use them to construct the half-spin representations of the Spin groups. At the infinitesimal level, if the Γ_a are as above, then

$$\Sigma_{ab} = \tfrac{1}{4}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)$$

define a representation of the Lie algebra $\mathrm{so}(s,t)$. But in fact, the spin group $\mathrm{Spin}(s,t)$ itself is contained in the Clifford algebra. It is convenient to introduce a little bit of notation. Consider first the *canonical automorphism* defined on the generators by $\Gamma_a \mapsto -\Gamma_a$. Under the action of this automorphism, the Clifford algebra decomposes into even and odd subalgebras:

$$\mathrm{C}\ell(s,t) = \mathrm{C}\ell(s,t)^{\mathrm{even}} \oplus \mathrm{C}\ell(s,t)^{\mathrm{odd}} ,$$

where $\mathrm{C}\ell(s,t)^{\mathrm{even}}$ (resp. $\mathrm{C}\ell(s,t)^{\mathrm{odd}}$) consists of real linear combinations of products of an even (resp. odd) number of Γ-matrices.

The Clifford algebra $\mathrm{C}\ell(s,t)$ contains several interesting subgroups. First of all we have the group of units $\mathrm{C}\ell(s,t)^\times$ consisting of all the invertible elements of $\mathrm{C}\ell(s,t)$. Clearly the group of units contains any other group inside $\mathrm{C}\ell(s,t)$. One of these groups is the Pin group $\mathrm{Pin}(s,t)$, defined to be the subgroup of $\mathrm{C}\ell(s,t)^\times$ generated by products of elements $v_i \in E$ whose norm is $q(v_i) = \pm 1$. The Spin group $\mathrm{Spin}(s,t)$ is the subgroup of $\mathrm{Pin}(s,t)$ consisting of those elements in $\mathrm{C}\ell(s,t)^{\mathrm{even}}$:

$$\mathrm{Pin}(s,t) \equiv \{v_1\, v_2 \cdots v_r | q(v_i) = \pm 1\}$$
$$\mathrm{Spin}(s,t) \equiv \{v_1\, v_2 \cdots v_{2k} | q(v_i) = \pm 1\} = \mathrm{Pin}(s,t) \cap \mathrm{C}\ell(s,t)^{\mathrm{even}} .$$

Therefore given a representation of $\mathrm{C}\ell(s,t)$ (resp. $\mathrm{C}\ell(s,t)^{\mathrm{even}}$) we automatically get a representation of $\mathrm{Pin}(s,t)$ (resp. $\mathrm{Spin}(s,t)$). Even if we start with an irreducible representation of $\mathrm{C}\ell(s,t)$, it may not remain irreducible as a representation of $\mathrm{Spin}(s,t)$; although if it is irreducible as a representation of $\mathrm{C}\ell(s,t)^{\mathrm{even}}$ it will remain irreducible under $\mathrm{Spin}(s,t)$. Irreducible representations of the Pin group (or of the Clifford algebra itself) are known are *pinor representations* whereas those of the even part of the Clifford algebra are known as *spinor representations*. A pinor representation consists of one or two spinor representations.

The Clifford algebras $\mathrm{C}\ell(s,t)$ have been classified. Their structure is periodic in $(s-t)$ with periodicity 8, as depicted in Table 1, where $\mathrm{Mat}_N(\Bbbk)$ denotes the algebra of $N \times N$ matrices with entries in $\Bbbk$. Notice that in general $\mathrm{C}\ell(s,t) \not\cong \mathrm{C}\ell(t,s)$. This means that strictly speaking the representations of the Clifford algebras depend on which metric we choose for the spacetime; that is, whether we use the mostly minus or mostly plus metrics.

$s-t \mod 8$	$\mathrm{C}\ell(s,t)$	N
$0,6$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{d/2}$
$2,4$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-2)/2}$
$1,5$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$

Table 1. Classification of Clifford algebras.

The classification is not hard to arrive at. It follows after an induction argument from the following two lemmas. Incidentally, the proof of the first lemma is very useful when it comes to constructing explicit realisations of Clifford algebras, as we will have ample opportunity to demonstrate in these notes.

Lemma 2. *The following "periods" hold:*

$$\mathrm{C}\ell(d,0) \otimes \mathrm{C}\ell(0,2) \cong \mathrm{C}\ell(0,d+2)$$
$$\mathrm{C}\ell(0,d) \otimes \mathrm{C}\ell(2,0) \cong \mathrm{C}\ell(d+2,0)$$
$$\mathrm{C}\ell(s,t) \otimes \mathrm{C}\ell(1,1) \cong \mathrm{C}\ell(s+1,t+1) .$$

Proof. To prove the first one, suppose that $\Gamma'_1, \Gamma'_2, \dots \Gamma'_d$ are Γ-matrices for $C\ell(d,0)$:

$$\Gamma'_a\Gamma'_b + \Gamma'_b\Gamma'_a = -2\delta_{ab}\mathbb{1} ;$$

and let Γ''_1 and Γ''_2 be Γ-matrices for $C\ell(0,2)$:

$$\Gamma''_a\Gamma''_b + \Gamma''_b\Gamma''_a = 2\delta_{ab}\mathbb{1} .$$

Then define the following Γ-matrices:

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_1\Gamma''_2 & \text{for } 1 \leq a \leq d, \\ \mathbb{1} \otimes \Gamma''_{a-d} & \text{for } a = d{+}1, d{+}2. \end{cases}$$

It is easy to show that they satisfy

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab}\mathbb{1} ,$$

whence they are Γ-matrices for $C\ell(0, d+2)$. The proof of the second one is entirely analogous. The last one is a little bit more involved, but it follows the same idea. Let $\Gamma'_1, \dots \Gamma'_s$ and $\tilde{\Gamma}'_1, \dots, \tilde{\Gamma}'_t$ be the Γ-matrices for $C\ell(s,t)$ and let Γ''_1 and $\tilde{\Gamma}''_1$ be the ones for $C\ell(1,1)$. Then define:

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_1\tilde{\Gamma}''_1 & \text{for } 1 \leq a \leq s, \\ \mathbb{1} \otimes \Gamma''_1 & \text{for } a = s{+}1; \end{cases}$$

$$\tilde{\Gamma}_a = \begin{cases} \tilde{\Gamma}'_a \otimes \Gamma''_1\tilde{\Gamma}''_1 & \text{for } 1 \leq a \leq t, \\ \mathbb{1} \otimes \tilde{\Gamma}''_1 & \text{for } a = t{+}1. \end{cases}$$

These are then Γ-matrices for $C\ell(s+1, t+1)$. □

Lemma 3. *The low-dimensional Clifford algebras are given by*

$$C\ell(1,0) \cong \mathbb{C} \qquad C\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$$
$$C\ell(2,0) \cong \mathbb{H} \qquad C\ell(1,1) \cong \mathrm{Mat}_2(\mathbb{R}) \qquad C\ell(0,2) \cong \mathrm{Mat}_2(\mathbb{R})$$

Proof. This follows from an explicit computation and will be reviewed below when we discuss the examples. □

Table 1 immediately teaches us about the pinor representations because the matrix algebra $\mathrm{Mat}_N(\mathbb{k})$ has a unique irreducible representation isomorphic to $\mathbb{k}^N$. Therefore we see, for example, that in even dimensions $d = s + t$, the Clifford algebra has a unique pinor representation $P(s,t)$ which is real of dimension $2^{d/2}$ if $s{-}t{=}0, 6 \pmod 8$ (Majorana) and quaternionic of dimension $2^{(d-2)/2}$ if $s{-}t{=}2, 4 \pmod 8$ (symplectic Majorana). Alternatively, if $d{=}s{+}t$ is odd, there are two inequivalent pinor representations, distinguished by the value of the *volume element* $\Gamma_{d+1} = \Gamma_1\Gamma_2\cdots\Gamma_d$, which in odd dimensions commutes with all the Γ-matrices. The possible values of Γ_{d+1} are determined as follows. Notice that Γ_{d+1} squares to $\pm\mathbb{1}$ depending on the signature:

$$\Gamma^2_{d+1} = (-1)^{(s-t+1)/2}\mathbb{1} . \tag{3}$$

Therefore for $s{-}t{=}1, 5 \pmod 8$, $\Gamma^2_{d+1} = -\mathbb{1}$ and there are two inequivalent complex pinor representations $P(s,t)$ and $\overline{P(s,t)}$ of complex dimension $2^{(d-1)/2}$: the Dirac pinors. They are distinguished by the value of Γ_{d+1}: it is i on $P(s,t)$ and $-i$ on $\overline{P(s,t)}$. Similarly, for $s{-}t{=}3 \pmod 8$, $\Gamma^2_{d+1} = +\mathbb{1}$, and there are two inequivalent quaternionic pinor representations $P(s,t)_\pm$ of quaternionic dimension $2^{(d-3)/2}$ and distinguished by the value of Γ_{d+1}: being ± 1 on $P(s,t)_\pm$. These are the symplectic Majorana pinors. Finally, for $s{-}t{=}7 \pmod 8$ there are two inequivalent real pinor representations $P(s,t)_\pm$ of real dimension $2^{(d-1)/2}$ and distinguished by the value of Γ_{d+1}: being ± 1 on $P(s,t)_\pm$. These are the Majorana pinors.

The situation for the spinor representations is similar but in a sense opposite. This is because of the fundamental isomorphisms:

$$\begin{aligned} \mathrm{C}\ell(s,t)^{\text{even}} &\cong \mathrm{C}\ell(s-1,t) \quad \text{for } s \geq 1 \\ \mathrm{C}\ell(s,t)^{\text{even}} &\cong \mathrm{C}\ell(t-1,s) \quad \text{for } t \geq 1 \end{aligned} \tag{4}$$

which together with Table 1 tell us the structure of the $\mathrm{C}\ell(s,t)^{\text{even}}$. These isomorphisms are again easy to prove. Suppose that $\Gamma_1, \ldots, \Gamma_s$ and $\Gamma_{s+1}, \ldots, \Gamma_{s+t}$ generate $\mathrm{C}\ell(s,t)$. Then if $s \geq 1$ we can define $\Gamma'_a = \Gamma_a \Gamma_1$ for $2 \leq a \leq d$. These matrices span $\mathrm{C}\ell(s,t)^{\text{even}}$ but at the same time are Γ-matrices for $\mathrm{C}\ell(s-1,t)$. On the other hand, if $t \geq 1$ we can define $\Gamma'_a = \Gamma_a \Gamma_d$ for $1 \leq a \leq d-1$. These matrices again span $\mathrm{C}\ell(s,t)^{\text{even}}$ but at the same time are Γ-matrices for $\mathrm{C}\ell(t-1,s)$.

The structure of the $\mathrm{C}\ell(s,t)^{\text{even}}$ is summarised in Table 2. Notice that now $\mathrm{C}\ell(s,t)^{\text{even}} \cong \mathrm{C}\ell(t,s)^{\text{even}}$, whence either choice of metric (mostly plus or mostly minus) yields the same type of representations. In a sense, the Spin group is a more intrinsic notion than the Clifford algebra, at least as far as the physics is concerned.

$s-t \mod 8$	$\mathrm{C}\ell(s,t)^{\text{even}}$	N
$1,7$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$
$3,5$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
$2,6$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
4	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-4)/2}$
0	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-2)/2}$

TABLE 2. Structure of the even subalgebras of a Clifford algebra.

From this table we immediately read that for odd dimensions there is a unique spinor representation $S(s,t)$ which is real of dimension $2^{(d-1)/2}$ for $s-t=1,7 \pmod 8$, and quaternionic of quaternionic dimension $2^{(d-3)/2}$ for $s-t=3,5 \pmod 8$. For even dimensions we have two inequivalent representations (Weyl spinors). This can again be understood by looking at the volume element or chirality Γ_{d+1}. In even dimensions, Γ_{d+1} anticommutes with the Γ-matrices, whence it commutes with $\mathrm{C}\ell(s,t)^{\text{even}}$. Therefore it must act like a scalar in any spinor representation. For even $d = s+t$, Γ_{d+1} obeys

$$\Gamma_{d+1}^2 = (-1)^{(s-t)/2}\mathbb{1} \ ,$$

whence if $s-t=2,6 \pmod 8$, $\Gamma_{d+1}^2 = -\mathbb{1}$ and if $s-t=0,4 \pmod 8$, $\Gamma_{d+1}^2 = +\mathbb{1}$. Therefore for $s-t=2,6 \pmod 8$ there are two inequivalent complex spinor representations $S(s,t)$ and $\overline{S(s,t)}$ of complex dimension $2^{(d-2)/2}$. They are distinguished by the value of Γ_{d+1}: it is i on $S(s,t)$ and $-i$ on $\overline{S(s,t)}$. Similarly, for $s-t=0 \pmod 8$ there are two inequivalent real spinor representations $S(s,t)_\pm$ of real dimension $2^{(d-2)/2}$ and distinguished by the value of Γ_{d+1}: being ± 1 on $S(s,t)_\pm$. These are the Majorana–Weyl spinors. Finally, for $s-t=4 \pmod 8$ there are two inequivalent quaternionic spinor representations $S(s,t)_\pm$ of quaternionic dimension $2^{(d-4)/2}$ and distinguished by the value of Γ_{d+1}: being ± 1 on $S(s,t)_\pm$. These are the symplectic Majorana–Weyl spinors.

The fundamental isomorphisms (4) allowed us to recover $\mathrm{C}\ell(s,t)^{\text{even}}$ from $\mathrm{C}\ell(s,t)$ or, equivalently, the spinor representations from the pinor representations. It turns out that one can go back, at least in odd dimensions. This is because $\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s,t)^{\text{even}} \oplus \Gamma_{d+1}\mathrm{C}\ell(s,t)^{\text{even}}$ and Γ_{d+1} commutes with all the Γ-matrices. Taking into account (3) we find that if we know $\mathrm{C}\ell(s,t)^{\text{even}}$ we may recover $\mathrm{C}\ell(s,t)$ as follows:

$$\mathrm{C}\ell(s,t) \cong \begin{cases} \mathrm{C}\ell(s,t)^{\text{even}} \otimes_{\mathbb{R}} \mathbb{C} & \text{for } s{-}t{=}1,5 \mod 8 \\ \mathrm{C}\ell(s,t)^{\text{even}} \otimes_{\mathbb{R}} (\mathbb{R}\oplus\mathbb{R}) & \text{for } s{-}t{=}3,7 \mod 8 \ . \end{cases}$$

where Γ_{d+1} gets sent to i in the first case and to $(1,-1)$ in the second case. This result tells us how to build the pinor representation from the spinor representation.

4. Complex Clifford algebras and the Majorana condition

The Clifford algebras of the previous section are real algebras: we are only allowed to take real linear combinations of products of Γ-matrices. As physicists we have no patience with restrictions of this kind and as a result we end up working with complex Clifford algebras. The immediate simplification is that all complex Clifford algebras of the same dimension are isomorphic:

$$\mathrm{C}\ell(s,t)\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\ell(s+t) \quad \text{for all } s,t, \tag{5}$$

whence the complex Clifford algebra $\mathbb{C}\ell(d)$ are classified according to Table 3.

$d \mod 2$	$\mathbb{C}\ell(d)$	N
0	$\mathrm{Mat}_N(\mathbb{C})$	$2^{d/2}$
1	$\mathrm{Mat}_N(\mathbb{C})\oplus\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$

Table 3. Structure of the complex Clifford algebras.

Just as in the real case, the classification table immediately reveals the structure of the pinor representations of the complex Clifford algebras. In even dimensions there is a unique complex representation of dimension $2^{d/2}$, whereas in odd dimensions there are two inequivalent complex pinor representations each of dimension $2^{(d-1)/2}$. These two representations can again be distinguished by the value of the volume element $\Gamma_{d+1} = \alpha\Gamma_1\Gamma_2\cdots\Gamma_d$, with the complex constant α chosen so that $\Gamma_{d+1}^2 = \mathbb{1}$, which as in the real case commutes with all Γ-matrices.

To understand the spinor representations we look at the even subalgebra of $\mathbb{C}\ell(d)$. From the isomorphisms (4) and (5) it follows that

$$\mathbb{C}\ell(d)^{\text{even}} \cong \mathbb{C}\ell(d-1) \ .$$

For convenience we record this in Table 4, which tells us that in odd dimensions there is a unique spinor representation of dimension $2^{(d-1)/2}$, whereas in even dimensions there are two spinor representations of dimension $2^{(d-2)/2}$, distinguished by the value of Γ_{d+1} which now commutes with $\mathbb{C}\ell(d)^{\text{even}}$.

$d \mod 2$	$\mathbb{C}\ell(d)^{\text{even}}$	N
0	$\mathrm{Mat}_N(\mathbb{C})\oplus\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
1	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$

Table 4. The even subalgebra of a complex Clifford algebra.

Therefore the advantage of complexifying the Clifford algebras is that their structure, and hence their representation theory, becomes more uniform. Of course this simplification comes at a price, precisely because we are not really interested in complex Clifford algebras and some machinery has to be developed in order to recover the finer structure that the complexification has hid. This machinery is precisely the traditional approach to Majorana spinors.

Let P denote a pinor representation of $C\ell(s,t)\otimes_{\mathbb{R}}\mathbb{C}$. As advertised, P possesses a nondegenerate hermitian form $\langle -,-\rangle$ which satisfies in addition the following "invariance" requirement:

$$\langle \Gamma_a \cdot \psi_1, \psi_2\rangle = \delta \langle \psi_1, \Gamma_a \cdot \psi_2\rangle \ ,$$

for all $\psi_1, \psi_2 \in P$ and all Γ_a, and where δ is a sign. Choosing a basis and letting $\boldsymbol{A}$ denote the hermitian matrix which represents $\langle -,-\rangle$, we can rewrite this condition as follows:

$$\overline{\Gamma}_a^t = \delta \boldsymbol{A}\cdot \Gamma_a \cdot \boldsymbol{A}^{-1} \ .$$

We now digress in order to prove that representations of Clifford algebras are unitarisable. This will follow from the result mentioned in Section 1 about finite groups, once we identify the Clifford algebra with (a quotient of) a group algebra. The group in question is the finite group generated by a choice of Γ-matrices. Let (Γ_a) satisfy the Clifford algebra (2). Let G denote the group generated by $\pm\Gamma_a$ and $\pm\mathbb{1}$ subject to the relations coming from (2). If $d = s+t$, then G has order 2^{d+1}. It is called the *Clifford group*. The Clifford algebra is almost the group algebra of G: it is the quotient of the group algebra of G by the relation $\mathbb{1} + (-\mathbb{1}) = 0$. In other words, not every representation of G will extend to a representation of the Clifford algebra. It will do so if and only if $-\mathbb{1}$ acts like -1. However every representation of the Clifford algebra does give rise to a representation of the Clifford group. In particular every such representation being unitarisable implies the same for the representations of the Clifford algebra. It follows from (2) that $\Gamma_a^2 = -\eta_{aa}\mathbb{1}$. This together with the fact that we can choose the Γ_a unitary, means that we can take them to obey

$$\overline{\Gamma}_a^t = \sigma_a \Gamma_a \qquad \text{where } \sigma_a = -\eta_{aa}.$$

Therefore we have that $\boldsymbol{A}$ satisfies

$$\sigma_a \Gamma_a \cdot \boldsymbol{A} = \delta \boldsymbol{A}\cdot\Gamma_a \ .$$

In other words $\boldsymbol{A}$ δ-commutes with the timelike Γ-matrices and δ-anticommutes with the spacelike Γ-matrices. We can solve these equations for $\boldsymbol{A}$ in terms of Γ-matrices and in fact we obtain up to a complex scalar multiple:

$$\boldsymbol{A} = \begin{cases} \Gamma_1\Gamma_2\cdots\Gamma_t & \text{for } \delta = (-1)^{t-1};\\ \Gamma_{t+1}\Gamma_{t+2}\cdots\Gamma_{t+s} & \text{for } \delta = (-1)^s. \end{cases}$$

The phase of the multiple can be fixed so that $\boldsymbol{A}$ is hermitian. In fact with the above choices:

$$\overline{\boldsymbol{A}}^t = \begin{cases} (-1)^{t(t-1)/2}\boldsymbol{A} & \text{for } \delta = (-1)^{t-1};\\ (-1)^{s(s+1)/2}\boldsymbol{A} & \text{for } \delta = (-1)^s. \end{cases}$$

We will not fix the phase explicitly, but simply note that it can be done. Whichever choice of δ we make and whichever hermiticity property $\boldsymbol{A}$ obeys, $\langle -,-\rangle$ is $\mathrm{Spin}(s,t)$-invariant:

$$\langle \Sigma_{ab}\cdot\psi_1, \psi_2\rangle = -\langle \psi_1, \Sigma_{ab}\cdot\psi_2\rangle \ .$$

We record for later use that

$$\boldsymbol{A}^2 = \begin{cases} (-1)^{t(t+1)/2}\mathbb{1} & \text{for } \delta = (-1)^{t-1};\\ (-1)^{s(s-1)/2}\mathbb{1} & \text{for } \delta = (-1)^s. \end{cases}$$

In particular we notice that $\boldsymbol{A}$ is nondegenerate. The choice of $\boldsymbol{A}$ (and hence δ) is inconsequential; although the first choice ($\boldsymbol{A} = \Gamma_1\Gamma_2\cdots\Gamma_t$) has been traditionally favoured by physicists.

Next we want to investigate whether this complex representation admits a real or quaternionic structure which is similarly invariant under the action of the Spin group. From Theorem 1 such a structure is equivalent to an invariant nondegenerate complex bilinear form B satisfying

$$B(\Gamma_a \cdot \psi_1, \psi_2) = \tau B(\psi_1, \Gamma_a \cdot \psi_2) \ ,$$

where τ is a sign and which in addition obeys $B(\psi_1, \psi_2) = \varepsilon B(\psi_2, \psi_1)$ depending on whether it is a real ($\varepsilon = +1$) or quaternionic ($\varepsilon = -1$) structure. In the chosen basis, B is represented by a matrix $\boldsymbol{B}$ which obeys

$$\Gamma_a^t = \tau \boldsymbol{B} \cdot \Gamma_a \cdot \boldsymbol{B}^{-1} \ . \tag{6}$$

As in the case of δ, the sign τ may be forced upon us; but any value of τ guarantees that B is $\mathrm{Spin}(s,t)$-invariant:

$$B(\Sigma_{ab} \cdot \psi_1, \psi_2) = -B(\psi_1, \Sigma_{ab} \cdot \psi_2) \ .$$

The signs ϵ and τ are not independent: they can be related to the signature (s,t) of the spacetime in the following way (a trick apparently due to Joel Scherk). First of all notice that from (6) it follows that

$$(\boldsymbol{B} \cdot \Gamma_a)^t = \varepsilon\tau \boldsymbol{B} \cdot \Gamma_a \ .$$

In turn this means that the matrices $\boldsymbol{B} \cdot \Gamma_{a_1}\Gamma_{a2} \cdots \Gamma_{a_p}$ for $1 \leq a_1 < a_2 < \cdots < a_p \leq d$ are also either symmetric or antisymmetric depending on ϵ, τ and s and t. But taking all those matrices together we span the complete matrix algebra in the appropriate dimension. Counting how many matrices are antisymmetric and comparing to the expected number $\frac{1}{2}n(n-1)$ we obtain a relation. We will not go into any more detail and instead refer the interested reader to [KT83].

Definition 4. *Let P be a pinor representation of a complexified Clifford algebra. We say that P is* Majorana *(resp.* symplectic Majorana*) if P admits a real (resp. quaternionic) structure $\mathcal{J}$. A pinor $\psi \in P$ is said to be* Majorana *if it satisfies $\mathcal{J}(\psi) = \psi$.*

To make contact with the traditional definition of Majorana spinors, simply notice the following. The condition $\mathcal{J}(\psi) = \psi$ means that for all pinors ψ', $B(\psi, \psi') = \langle \mathcal{J}(\psi), \psi' \rangle = \langle \psi, \psi' \rangle$; or in the chosen basis,

$$\psi^t \cdot \boldsymbol{B} \cdot \psi' = \overline{\psi}^t \cdot \boldsymbol{A} \cdot \psi' \ .$$

Since this is true for all $\psi' \in P$ we see that $\psi \in P$ is Majorana if and only if

$$\psi^t \cdot \boldsymbol{B} = \overline{\psi}^t \cdot \boldsymbol{A} \ .$$

The right-hand side of this equation defines the Dirac conjugate of the pinor ψ whereas the left-hand side defines the Majorana conjugate. Then a pinor ψ is Majorana if and only if its Dirac and Majorana conjugates agree—the traditional definition. Of course, traditionally $\boldsymbol{B}$ is the charge conjugation matrix and is usually written $\boldsymbol{C}$.

If P possesses a quaternionic structure, we cannot impose the Majorana condition $\mathcal{J}(\psi) = \psi$, because $\mathcal{J}^2 = -\mathbb{1}$ forces $\psi = 0$; but all is not lost. We can consider spinors with flavour in some quaternionic representation V of some group G. Then from Corollary 1, we know that the representation $P \otimes V$ of $\mathrm{Spin}(s,t) \times G$ is real, with real structure $\mathcal{J}_\otimes$ equal to the tensor product of the quaternionic structures of P and V. It then makes sense to impose the condition $\mathcal{J}_\otimes(\psi) = \psi$ on a pinor $\psi \in P \otimes V$. Pinors satisfying this reality condition are known as symplectic Majorana spinors.

5. Examples

To illustrate the remarks in the previous two sections, let us now work out some examples. We start in one dimension and move our way up to the currently fashionable case of twelve dimensions, calling at 2, 3, 4, 6, 10 and 11 dimensions along the way. The first two cases will also serve to prove Lemma 3.

One dimension. This one is easy. We either have $\Gamma_1^2 = 1$ in the case of $C\ell(0,1)$ or $\Gamma_1^2 = -1$ in the case of $C\ell(1,0)$. In the latter case, Γ_1 is a complex structure and $C\ell(1,0) \cong \mathbb{C}$, whereas in the former $C\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$. This proves the first line of Lemma 3.

Two dimensions. Together with the somewhat trivial case in one dimension, the two-dimensional Clifford algebras are the basic building blocks of the theory, so it pays to understand them well. We have to consider three signatures: $(2,0)$, $(1,1)$ and $(0,2)$.

We start with $(1,1)$ and adopt the physics convention of numbering the Γ-matrices starting at 0. We need two matrices satisfying $\Gamma_0^2 = +\mathbb{1}$ and $\Gamma_1^2 = -\mathbb{1}$. A possible choice is

$$\Gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The volume element $\Gamma_3 = \Gamma_0 \Gamma_1$ is given by

$$\Gamma_3 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Because the Γ-matrices are real and 2×2, we see that $C\ell(1,1) \cong \mathrm{Mat}_2(\mathbb{R})$ in agreement with Lemma 3. The even subalgebra is generated by $\mathbb{1}$ and Γ_3, which here squares to $\mathbb{1}$, hence the even subalgebra is that of diagonal matrices: $C\ell(1,1)^{\mathrm{even}} \cong \mathbb{R} \oplus \mathbb{R}$ in agreement with Table 2. The pinors are Majorana (2-component, real) whereas the spinors are Majorana–Weyl (1-component, real).

We continue with the signature $(0,2)$. In this case we need matrices Γ_1 and Γ_2 which anticommute and square to $\mathbb{1}$. A possible choice is

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Again these matrices are real, whence $C\ell(0,2) \cong \mathrm{Mat}_2(\mathbb{R})$ in agreement with Lemma 3. The volume element is now:

$$\Gamma_3 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

which squares to $-\mathbb{1}$. This means that the even subalgebra, being generated by $\mathbb{1}$ and Γ_3 is now isomorphic to $\mathbb{C}$. Therefore, although the pinors are again Majorana (2-component, real), the spinors are simply Weyl (1-component, complex). Weyl spinors of opposite chirality are complex conjugate.

Finally we discuss the euclidean signature $(2,0)$. A possible choice for Γ-matrices is given by

$$\Gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The matrices are not real. In fact, how could they? The Clifford algebra is generated by two anticommuting complex structures, hence it has to be the quaternion algebra $\mathbb{H}$. Indeed an explicit isomorphism is: $(1,i,j,k) = (\mathbb{1}, \Gamma_1, \Gamma_2, \Gamma_3)$, with $\Gamma_3 = \Gamma_1\Gamma_2$ the volume element:

$$\Gamma_3 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} .$$

The even subalgebra is generated by $\mathbb{1}$ and Γ_3, which is a complex structure, hence $C\ell(2,0)^{\text{even}} \cong \mathbb{C}$, in agreement with the table. Hence in two-dimensions, euclidean pinors are Dirac (2-component complex, but secretly a quaternion), whereas spinors are as in $(0,2)$ case: complex Weyl. According to Theorem 1 if the representation is quaternionic, there is an antisymmetric bilinear form lurking around somewhere. In the above realisation, it is given by Γ_2.

Three dimensions. We will only look at the euclidean case. In this case, the spin group is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and the spinor representation is the complex two-dimensional representation. We know since infancy that $\mathrm{SU}(2)$ has only one spin-$\frac{1}{2}$ representation up to equivalence, hence this cannot be the end of the story. In other words, if the representation were truly complex, then the complex conjugate representation would furnish us with a second inequivalent spin-$\frac{1}{2}$ representation. According to Table 2 we see that $C\ell(3,0)^{\text{even}} \cong \mathrm{Mat}_1(\mathbb{H}) = \mathbb{H}$, whence the representation is quaternionic. According to Theorem 1, a quaternionic structure is equivalent to an antisymmetric complex bilinear form and, indeed, $\mathrm{SU}(2)$ leaves invariant the ϵ-symbol. To find the explicit expression for the quaternionic structure, we follow the procedure indicated in Section 2; that is, we use (1) for $\boldsymbol{J}$. We need to determine $\boldsymbol{A}$ and $\boldsymbol{B}$. In this case, $\mathrm{SU}(2)$ leaves invariant the canonical hermitian metric on $\mathbb{C}^2$ which is represented by $\boldsymbol{A} = \mathbb{1}$. Similarly, $\boldsymbol{B}$ is given by the ϵ-symbol; that is, $\boldsymbol{B} = i\sigma_2 \in \mathrm{Mat}_2(\mathbb{C})$. Therefore, using (1), we find $\boldsymbol{J} = -i\sigma_2$ which obeys $\boldsymbol{J}^2 = -\mathbb{1}$ as expected.

Four dimensions. We now come to the ur-example: Minkowski spacetime in the mostly minus metric. The associated Clifford algebra is $C\ell(1,3) \cong \mathrm{Mat}_4(\mathbb{R})$. The pinor space is therefore isomorphic to $\mathbb{R}^4$. These are the Majorana spinors. How about the spinors? From Table 2 we see that $C\ell(1,3)^{\text{even}} \cong \mathrm{Mat}_2(\mathbb{C})$, which can be understood as follows. For this signature the volume element Γ_5 is a complex structure, and $C\ell(1,3)^{\text{even}}$ is defined as the subalgebra of $C\ell(1,3)$ which commutes with Γ_5. In other words, those matrices in $\mathrm{Mat}_4(\mathbb{R})$ which commute with the complex structure: but these are precisely $\mathrm{Mat}_2(\mathbb{C}) \subset \mathrm{Mat}_4(\mathbb{R})$. At any rate, the spinors now come in two inequivalent complex two-dimensional representations. These correspond to the so-called dotted and undotted $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}(1,3)$ spinors: the $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ representations; which are indeed inequivalent. We can give an explicit realisation for the Γ-matrices by exploiting Lemma 2. We find that

$$\Gamma_0 = \mathbb{1} \otimes i\sigma_2 \quad \Gamma_1 = \sigma_1 \otimes \sigma_3 \quad \Gamma_2 = \sigma_3 \otimes \sigma_3 \quad \Gamma_3 = \mathbb{1} \otimes \sigma_1 \ ,$$

which, being manifestly real, provide a Majorana representation for the Γ-matrices. In the spirit of Theorem 1 we ought to look for a symmetric bilinear form B which is responsible for having a real structure. What makes the Majorana representation special is that since the Γ-matrices are already real, the matrix representing the bilinear form is the identity.

How about the mostly plus metric? From the table we read $C\ell(3,1) \cong \mathrm{Mat}_2(\mathbb{H})$, whence there is a unique quaternionic two-dimensional pinor representation. This is the familiar Dirac spinor, although the quaternionic structure is seldom emphasised (but see below). How about the spinors in this case? There is no difference between 3+1 and 1+3—there never is at the level of Spin groups—since again $C\ell(3,1)^{\text{even}} \cong \mathrm{Mat}_2(\mathbb{C})$. An explicit representation for the Γ-matrices is given by:

$$\Gamma_0 = \mathbb{1} \otimes \sigma_1 \quad \Gamma_1 = -i\sigma_1 \otimes \sigma_3 \quad \Gamma_2 = -i\sigma_2 \otimes \sigma_3 \quad \Gamma_3 = \mathbb{1} \otimes i\sigma_2 \ ,$$

which is unitarily related to the more familiar one:

$$\tilde{\Gamma}_0 = \mathbb{1} \otimes \sigma_1 \quad \tilde{\Gamma}_1 = \sigma_1 \otimes i\sigma_2 \quad \tilde{\Gamma}_2 = \sigma_2 \otimes i\sigma_2 \quad \tilde{\Gamma}_3 = \sigma_3 \otimes i\sigma_2 \ .$$

According to Theorem 1, a quaternionic structure is equivalent to an invariant antisymmetric bilinear form in the pinor space thought of as a four-dimensional complex vector space. After choosing a basis for the pinors, this form is represented by a 4×4 complex matrix $\boldsymbol{B}$: $B(\psi_1, \psi_2) = \boldsymbol{\psi}_1^t \cdot \boldsymbol{B} \cdot \boldsymbol{\psi}_2$ for all pinors ψ_1, ψ_2. Invariance under the Clifford algebra is equivalent to $B(\Gamma_a \psi_1, \Gamma_a \psi_2) = B(\psi_1, \psi_2)$. Which means that $\Gamma_a^t \cdot \boldsymbol{B} \cdot \Gamma_a = \boldsymbol{B}$. From the above explicit realisation, we see that Γ_0 and Γ_1 are symmetric, whereas Γ_2 and Γ_3 are antisymmetric. Therefore $\boldsymbol{B}$ must anticommute with Γ_1 and commute with Γ_0, Γ_2 and Γ_3. In other words, $\boldsymbol{B} \propto \Gamma_0 \Gamma_2 \Gamma_3$. One checks that $\boldsymbol{B}^t = -\boldsymbol{B}$ as expected. In the above basis $\boldsymbol{B} = -i\sigma_1 \otimes \mathbb{1}$.

The four-dimensional case teaches us something: strictly speaking there are no Majorana spinors in 3+1 dimensions. There are, however, pseudo-Majorana spinors – a nebulous concept best kept undisturbed. Let us simply remark that pseudo-Majorana spinors exist for $\mathrm{C}\ell(s,t)$ if and only if Majorana spinors exist for $\mathrm{C}\ell(t,s)$.

Let us now look at the euclidean signature $(4,0)$. From the table $\mathrm{C}\ell(4,0) \cong \mathrm{Mat}_2(\mathbb{H})$. We can think of $\mathrm{Mat}_2(\mathbb{H}) \subset \mathrm{Mat}_4(\mathbb{C})$ as the subalgebra commuting with a quaternionic structure. An explicit realisation is given by

$$\Gamma_1 = -\sigma_1 \otimes i\sigma_3 \quad \Gamma_2 = -\sigma_3 \otimes i\sigma_3 \quad \Gamma_3 = \mathbb{1} \otimes i\sigma_1 \quad \Gamma_4 = \mathbb{1} \otimes i\sigma_2 \ .$$

In this realisation all Γ-matrices are antihermitian, and symmetric except for Γ_4 which, being real, is antisymmetric. This means that the antisymmetric bilinear form guaranteed by Theorem 1 is represented in the above realisation by Γ_4.

Finally we consider $(2,2)$. From the table, $\mathrm{C}\ell(2,2) \cong \mathrm{Mat}_4(\mathbb{R})$ again, whence the pinors are Majorana. An explicit realisation is given by

$$\Gamma_1 = -\sigma_1 \otimes \sigma_3 \quad \Gamma_2 = \mathbb{1} \otimes \sigma_1 \quad \Gamma_3 = -i\sigma_2 \otimes \sigma_3 \quad \Gamma_4 = \mathbb{1} \otimes i\sigma_2 \ ,$$

which is again a Majorana realisation, since all the Γ-matrices are real. Spinors are Majorana–Weyl (2-component real), because $\mathrm{C}\ell(2,2)^{\text{even}} \cong \mathrm{Mat}_2(\mathbb{R}) \oplus \mathrm{Mat}_2(\mathbb{R})$.

Six dimensions. Moving up in dimensions, we next stop to consider six-dimensional Minkowski spacetime. In this case there is no difference between the two metrics, since $s-t=4 \pmod 8 = -4 \pmod 8$. From the classification we see that $\mathrm{C}\ell(5,1) \cong \mathrm{Mat}_4(\mathbb{H})$, whence there is a unique pinor representations, which is quaternionic of dimension 4. These are the expected 8-component Dirac spinors but with an under-emphasised quaternionic structure. As for spinors there are two inequivalent spinor representations, distinguished by chirality. They are quaternionic of dimension 2. These will be the building blocks for the symplectic Majorana–Weyl spinors we're after. It follows from Lemma 2 that $\mathrm{C}\ell(5,1) \cong \mathrm{C}\ell(4,0) \otimes \mathrm{C}\ell(1,1)$, whence if $\{\hat{\Gamma}_a\}$ are Γ-matrices for $\mathrm{C}\ell(4,0)$, then we have

$$\begin{aligned} &\Gamma_0 = \mathbb{1} \otimes \sigma_1 && \Gamma_1 = -\hat{\Gamma}_1 \otimes \sigma_3 && \Gamma_2 = -\hat{\Gamma}_2 \otimes \sigma_3 \\ &\Gamma_3 = -\hat{\Gamma}_3 \otimes \sigma_3 && \Gamma_4 = -\hat{\Gamma}_4 \otimes \sigma_3 && \Gamma_5 = \mathbb{1} \otimes i\sigma_2 \ . \end{aligned}$$

Using the explicit representation of $\mathrm{C}\ell(4,0)$ that we found above we see that the antisymmetric bilinear form responsible for the quaternionic structure is given by $\Gamma_0 \Gamma_4 \Gamma_5$, which in this realisation is $-\mathbb{1} \otimes i\sigma_2 \otimes \mathbb{1}$. As expected, it is antisymmetric.

Let us now consider euclidean space. From the classification we see that $\mathrm{C}\ell(6,0) \cong \mathrm{Mat}_8(\mathbb{R})$ whence pinors are 8-component Majorana spinors. As for spinors we notice that $\mathrm{C}\ell(6,0)^{\text{even}} \cong \mathrm{Mat}_4(\mathbb{C})$ so that they are four-component complex. Indeed there is an accidental isomorphism $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ under which the two spinor representations become the fundamental representation of $\mathrm{SU}(4)$ and its conjugate representation. Let us call these representations $S_\pm$ with $S_+ \cong \bar{S}_-$. Dirac spinors transform according to the reducible complex representation $S_+ \oplus S_-$, which is clearly the complexification of a real representation under which the Majorana spinors transform.

Ten dimensions. Going on to bigger and better things, let's look at ten dimensional Minkowski spacetime. As in six-dimensions there is no difference between the two possible metrics as $C\ell(9,1) \cong C\ell(1,9) \cong \mathrm{Mat}_{32}(\mathbb{R})$. In other words, the pinor representation is a 32-component *real* spinor: the ten-dimensional Majorana spinor. According to Table 2, there are two inequivalent spinor representations, which are real and 16-dimensional. These are the positive and negative chirality Majorana–Weyl spinors. We can use Lemma 2 in order to find an explicit realisation for the Γ-matrices. According to the lemma, $C\ell(9,1) \cong C\ell(4,0) \otimes C\ell(0,2) \otimes C\ell(2,0) \otimes C\ell(1,1)$. Using the realisations found above for these algebras one comes up with the following:

$$\begin{aligned}
\Gamma_0 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes 1 & \Gamma_9 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes i\sigma_2 \\
\Gamma_8 &= -\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes i\sigma_2 \otimes \sigma_3 & \Gamma_7 &= -\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes i\sigma_1 \otimes \sigma_3 \\
\Gamma_6 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 \otimes i\sigma_3 \otimes \sigma_3 & \Gamma_5 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_1 \otimes i\sigma_3 \otimes \sigma_3 \\
\Gamma_4 &= -\mathbb{1} \otimes i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_3 \otimes \sigma_3 & \Gamma_3 &= -\mathbb{1} \otimes i\sigma_1 \otimes i\sigma_2 \otimes i\sigma_3 \otimes \sigma_3 \\
\Gamma_2 &= \sigma_3 \otimes i\sigma_3 \otimes i\sigma_2 \otimes i\sigma_3 \otimes \sigma_3 & \Gamma_1 &= \sigma_1 \otimes i\sigma_3 \otimes i\sigma_2 \otimes i\sigma_3 \otimes \sigma_3 \ .
\end{aligned}$$

The symmetric bilinear form is given by $\boldsymbol{B} = \Gamma_4\Gamma_5\Gamma_6\Gamma_7$, which in this realisation is given by $\boldsymbol{B} = \mathbb{1} \otimes i\sigma_2 \otimes \mathbb{1} \otimes i\sigma_2 \otimes \mathbb{1}$, which is real and symmetric.

Eleven dimensions. Now to be really modern, we look at eleven and twelve-dimensions. In eleven dimensions and as far as the pinor representations are concerned, it does matter which metric we choose, because $C\ell(10,1) \cong \mathrm{Mat}_{32}(\mathbb{C})$ but $C\ell(1,10) \cong \mathrm{Mat}_{32}(\mathbb{R}) \oplus \mathrm{Mat}_{32}(\mathbb{R})$. In other words, in the mostly plus metric, the pinors are 32-component complex; whereas in the mostly minus metric, the pinors are real and have 32 components. This is similar to what happened in four dimensions. The mostly minus metric admits Majorana spinors whereas the mostly plus metric admits pseudo Majorana spinors. Of course, as far as the spinor representation is concerned, there is only one and it is 32-component real. This is the spinorial representation under which the supercharge in eleven-dimensional supergravity transforms. Upon dimensional reduction to ten-dimensions, it yields two Majorana–Weyl spinors: one of each chirality. (More about this below.)

Twelve dimensions. Finally, we come to twelve dimensions and signature $(10,2)$—the choice of metric being irrelevant again. From the classification we see that $C\ell(10,2) \cong \mathrm{Mat}_{64}(\mathbb{R})$ and there is a unique real 64-dimensional pinor representation. This is a Majorana spinor in 10+2. On the other hand this representation breaks up into two spinor representations of different chiralities: the 32-component Majorana–Weyl spinors in 10+2.

...and back! Having come all the way up to twelve dimensions, it is hard not to lose one's balance and come tumbling back down again. We shall do so under the fancy guise of dimensional reduction. The main observation is that $C\ell(s,t)$ contains $C\ell(s',t')$ as a subalgebra whenever $s' \leq s$ and $t' \leq t$. We can therefore start with a (s)pinor of $C\ell(s,t)$ and see how it breaks up under $C\ell(s',t')$ for successively smaller values of $s'+t'$. In able hands, this procedure teaches us quite a lot about the interrelations between different supersymmetric theories.

It may be easier to read this subsection while staring at Table 5. Being "F-isically" motivated, we will start with a spinor of $C\ell(2,10)$, the choice of chirality being irrelevant. We mean, of course, a spinor of the even subalgebra, but we trust this can safely remain implicit. As we have just seen this is 32-component and real. A supersymmetry generator in 2+10 dimensions would be such a spinor and we would call such supersymmetry algebra $N{=}(1,0)$ or $N{=}(0,1)$ depending on the choice of chirality. Because $C\ell(2,10)^{\mathrm{even}} \cong C\ell(1,10)$, this very spinor becomes one

$s+t$	d	$s-t \mod 8$	Pinors	Spinors
$2+10$	12	0	$\mathbb{R}^{64}$	$\mathbb{R}^{32} \oplus \mathbb{R}^{32}$
$1+10$	11	7	$\mathbb{R}^{32} \oplus \mathbb{R}^{32}$	$\mathbb{R}^{32}$
$1+9$	10	0	$\mathbb{R}^{32}$	$\mathbb{R}^{16} \oplus \mathbb{R}^{16}$
$1+8$	9	1	$\mathbb{C}^{16}$	$\mathbb{R}^{16}$
$1+7$	8	2	$\mathbb{H}^{8}$	$\mathbb{C}^{8}$
$1+6$	7	3	$\mathbb{H}^{4} \oplus \mathbb{H}^{4}$	$\mathbb{H}^{4}$
$1+5$	6	4	$\mathbb{H}^{4}$	$\mathbb{H}^{2} \oplus \mathbb{H}^{2}$
$1+4$	5	5	$\mathbb{C}^{4}$	$\mathbb{H}^{2}$
$1+3$	4	6	$\mathbb{R}^{4}$	$\mathbb{C}^{2}$
$1+2$	3	7	$\mathbb{R}^{2} \oplus \mathbb{R}^{2}$	$\mathbb{R}^{2}$
$1+1$	2	0	$\mathbb{R}^{2}$	$\mathbb{R} \oplus \mathbb{R}$
$1+0$	1	1	$\mathbb{C}$	$\mathbb{R}$

TABLE 5. Dimensional reduction data.

of the two inequivalent pinors of $C\ell(1,10)$ – which one we get depending on the original choice of chirality in 2+10 dimensions. The supercharge in N=1 supergravity in 1+10 dimensions transforms under this pinor representation.

As mentioned before in the subsection on eleven dimensions, under $C\ell(1,9)$ this pinor representation becomes a pair of spinors: one of each chirality. This corresponds to N=$(1,1)$ supergravity (or type IIA) in 1+9 dimensions. Chiral supergravities are of course also possible: N=$(2,0)$ (or type IIB); but they don't seem to arise as dimensional reduction. In 1+9 dimensions we also have N=$(1,0)$ supersymmetric Yang–Mills. This can be obtained from 2+10 but this is sadly beyond the scope of these notes at present.

We now go briefly though the other dimensions of interest. In 1+5 dimensions supersymmetries are Weyl spinors; so that in principle we have the possibility of having chiral supersymmetries. In 1+3 dimensions, supersymmetries are Majorana spinors. In 1+1 dimensions supersymmetries are again Majorana–Weyl spinors and we once again have the possibility of chiral supersymmetry. Finally in 1+0 dimensions supersymmetries are Majorana. Every chiral supersymmetry in 1+1 dimensions gives rise to a supersymmetry in 1+0 dimensions; whereas each supersymmetry in 1+3 dimensions gives rise to $(2,2)$ supersymmetry in 1+1 dimensions. In turn every chiral supersymmetry in 1+5 dimensions gives rise to 2 supersymmetries in 1+3 dimensions; and finally each chiral supersymmetry in 1+9 dimensions gives rise to $(1,1)$ supersymmetry in 1+5 dimensions. If we use the notation $(p,q)_{s+t}$ or N_{s+t} to refer to (p,q) or N supersymmetry in $s+t$ dimensions, we have the following chains of dimensional reductions:

$$\begin{aligned} 1_{1+10} \rightsquigarrow (1,1)_{1+9} \rightsquigarrow (2,2)_{1+5} \rightsquigarrow 8_{1+3} \rightsquigarrow (16,16)_{1+1} &\rightsquigarrow 32_{1+0} \\ (1,0)_{1+9} \rightsquigarrow (1,1)_{1+5} \rightsquigarrow 4_{1+3} \rightsquigarrow (8,8)_{1+1} &\rightsquigarrow 16_{1+0} \\ (1,0)_{1+5} \rightsquigarrow 2_{1+3} \rightsquigarrow (4,4)_{1+1} &\rightsquigarrow 8_{1+0} \\ 1_{1+3} \rightsquigarrow (2,2)_{1+1} &\rightsquigarrow 4_{1+0} \\ (1,0)_{1+1} &\rightsquigarrow 1_{1+0} \end{aligned}$$

$d=12$	♦	◊	■	□	♦	◊	■	□	♦	◊	■	□	♦
11	◊		□		◊		□		◊		□		
10	◊	■	□	♦	◊	■	□	♦	◊	■	□		
9		□		◊		□		◊		□			
8	■	□	♦	◊	■	□	♦	◊	■				
7	□		◊		□		◊						
6	□	♦	◊	■	□	♦	◊						
5		◊		□		◊							
4	♦	◊	■	□	♦								
3	◊		□										
2	◊	■	□										
1		□											
$t=$	0	1	2	3	4	5	6	7	8	9	10	11	12

□ Majorana ◊ symplectic Majorana
■ Majorana–Weyl ♦ symplectic Majorana–Weyl

TABLE 6. Table of spinor types as a function of (d,t).

$d=12$	♦	□	■	◊	♦	□	■	◊	♦	□	■	◊	♦
11		□		◊		□		◊		□		◊	
10	□	■	◊	♦	□	■	◊	♦	□	■	◊		
9	□		◊		□		◊		□				
8	■	◊	♦	□	■	◊	♦	□	■				
7		◊		□		◊		□					
6	◊	♦	□	■	◊	♦	□						
5	◊		□		◊								
4	♦	□	■	◊	♦								
3		□		◊									
2	□	■	◊										
1	□												
$s=$	0	1	2	3	4	5	6	7	8	9	10	11	12

□ Majorana ◊ symplectic Majorana
■ Majorana–Weyl ♦ symplectic Majorana–Weyl

TABLE 7. Table of spinor types as a function of (d,s).

Summary. From the Tables we can see at a glance in which spacetimes we have Majorana spinors (i.e., real pinors), symplectic Majorana spinors (i.e., quaternionic pinors), and for which we have Majorana–Weyl spinors (i.e., real spinors) and symplectic Majorana–Weyl spinors (i.e., quaternionic spinors). We have Majorana spinors for $s{-}t{=}0,6,7 \pmod 8$, whereas only for $s{-}t{=}0 \pmod 8$ do we have Majorana–Weyl spinors. Similarly, only for $s{-}t{=}2,3,4 \pmod 8$ do we have symplectic Majorana spinors whereas for symplectic Majorana–Weyl spinors we have to restrict ourselves to $s{-}t{=}4 \pmod 8$. We could repeat this paragraph interchanging s and t and inserting a 'pseudo' before every spinor type and still arrive at a true result, but we will resist the temptation. Some of these results are summarised in Tables 6 and 7. In Table 6, t refers to the number of timelike coordinates in a mostly plus metric; whereas in Table 7, s denotes the number of timelike coordinates in a mostly minus metric. Both tables are consistent with the notation in these notes—I have simply included them both for convenience. Notice that for even dimensions we always have either Majorana or symplectic Majorana spinors.

It is worth pointing out that according to Table 2 it is sometimes possible to reduce the size of the spinors (that is, as representations of the Spin group) in ways that are not reflected in the above tables. For example, when $s{-}t{=}1 \pmod 8$, it is possible to have real spinors, even though pinors are complex. Holonomy cognoscenti will immediately recall that the unique half-spin representation of Spin(9) is real and sixteen-dimensional; whereas pinors in 9+0 dimensions are complex sixteen-dimensional. Evidently the real and imaginary parts of the pinors transform into each other under the even subalgebra $C\ell(9,0)^{\text{even}}$, whereas the individual Γ-matrices mix them. In other words, there is a $C\ell(9,0)^{\text{even}}$-invariant real structure in the pinor representation, which however is not invariant under the full Clifford algebra. A similar situation arises for $s{-}t{=}5 \pmod 8$, when there is a $C\ell(s,t)^{\text{even}}$-invariant quaternionic structure. Spinors in these signatures are thus quaternionic, whereas the pinors remain complex.

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